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Construction of Class 2 Graphs with Maximum Vertex Degree 3

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A new method is developed for constructing graphs with maximum vertex degree 3 and chromatic index 4. In particular an infinite family of edge-critical graphs with an even number of vertices is constructed; this disproves the Critical Graph Conjecture.

1. INTRODUCTION

The well-known theorem of Vizing [17] states that the chromatic index of any finite graph with maximum degree r is equal either to r or to $r+1$. In the former case the graph G is said to be of class 1, $\mathrm{cl}(G)=1$, in the latter case $\mathrm{cl}(G)=2$.

The classification problem, i.e., the problem of deciding which graph belongs to which class is very interesting and exceedingly difficult. Until recently it was difficult even to construct a sufficiently representative system of class 2 graphs with maximum vertex degree 3. To the author's knowledge the history of the search for such graphs is as follows.

In the papers by Blanuša [4] and by Descartes [5] a simple method was introduced for reducing the problem for some graphs to smaller graphs. In fact a reducible graph may be described as one which contains a subset $V'(|V'|>1)$ $(|V-V'|>1)$ of the vertex-set such that all degrees of vertices belonging to V' are equal to 3 and the number of edges which connect V' with the rest of the graph does not exceed 3. Among regular graphs only cyclically 4-edge-connected graphs are irreducible.

In [7] Gardner proposed the term "snarks" for those regular graphs of class 2 which are irreducible. Until 1973 only four snarks had been discovered. They were found by Petersen (the famous Petersen graph) [15], Blanuša [4], Descartes [5] and Szekeres [16], respectively.

In [1, 11] an operation was described which constructs a new snark from

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a pair of others. It permitted the generation of an infinite system of snarks starting from the Petersen graph. All the above-mentioned snarks belong to this system which we call the BDS-system, following Isaacs [11].

A wide family of class 2 graphs (not necessarily regular) was described by Jakobsen [13], who showed that the well-known Hajós-union [10] applied to graphs of class 2 satisfying certain conditions, results in a new graph of class 2.

It is clear that in the general case a nonregular graph of class 2 allows one to construct many snarks by trivially adding new vertices and edges. On the other hand it is usually difficult to find those edges and vertices whose deletion from a snark yields class 2 graphs. In particular some cyclically 5-edge-connected graphs of class 2 and some snarks from the BDS-system (never cyclically 5-edge-connected graphs) may be produced from the same graph in Jakobsen's system.

A remarkable sequence $\{I_k\}$ of snarks was discovered by Isaacs in the above-mentioned paper [11]. In the same paper Isaacs described a further snark and called it a double-star. Later a new wide system of snarks was also described by Isaacs in [12] (Loupekhine snarks). None of these snarks can be obtained by Jakobsen's method.

It should be noted that the sequence $\{I_k\}$ was independently discovered (though never published) by E. Ya. Grinberg and reported at the Symposium on Graph Theory held in Vaivary (Latvia) in 1972.

In the present paper we provide a new and very simple method of constructing class 2 graphs. In particular an infinite sequence of irreducible edge-critical graphs with an even number of vertices is constructed, disproving the Critical Graph Conjecture. Let us note that the smallest graph of this sequence turns out to be a subgraph of the smallest Loupekhine snark.

2. Terminology

For the purpose of this paper it is convenient to modify the usual definition of an undirected graph by introducing the notion of a semiedge. A graph G is a triple $\{V(G), E(G), S(G)\}$, where $V(G)$ is a finite set of elements called vertices, $E(G)$ is a set of distinct unordered pairs of distinct elements of $V(G)$ called edges and $S(G)$ is a finite set of elements called semiedges which map into $V(G)$.

All usual notions either remain unchanged or need minimal and evident corrections. Thus, the degree of a vertex is defined to be the number of edges incident to it; chains beginning or ending with semiedges are permitted; a colouring of a graph G is a function defined on the set $E(G) \cup S(G)$ and satisfying the condition that all edges and semiedges incident to the same vertex have different colours.

Semiedges are in effect edges incident to a single vertex which constitute their one end.

An edge u (semiedge v) incident to vertices x and y (resp. a vertex x) will be denoted by (x, y) (resp. by (x)).

Let (x) and (y) be two semiedges $(x \neq y)$. The identification of (x) and (y) consists of the replacement of these semiedges by the edge (x, y).

If an edge $u = (x, y)$ is removed from G then u is replaced by two new semiedges (x) and (y); the resulting graph is denoted by G_u.

If a vertex x is removed from graph G then all semiedges (x) are removed and each edge (x, y) is substituted by the semiedge (y); the resulting graph is denoted by G_x.

Throughout the paper the following conditions are satisfied:

(i) All vertex degrees are at most 3.

(ii) All colourings use only three colours.

(iii) The total number of edges and semiedges incident to the same vertex is equal to 3.

3. Graphs and Cells; Rules of Construction

This section contains two rules for constructing class 2 graphs from certain graphs called cells. It also contains a description of the methods for generating cells (Lemma 3, Remarks I–IV).

Definition 1. A graph G with some fixed set M of disjoint pairs of semiedges is called a cell; M is called a d-set; the cardinality of M is called the rank of the cell and denoted by $\operatorname{rank}(D)$.

Definition 2. Let D' and D'' be cells with d-sets M' and M'', respectively, let $N' = \{(u_i', v_i')\} \subset M'$, $N'' = \{(u_i'', v_i'')\} \subset M''$ and let $k = |N'| = |N''|$. In the union $D' \cup D''$ let us identify semiedges u_i' with u_i'', v_i' with v_i'' $(i = 1, \ldots, k)$. The resulting graph D with the d-set $(M' \backslash N') \cup (M'' \backslash N'')$ will be called a junction of D' and D'' with respect to N' and N'', or simply a junction. A junction will be called complete if $N' = M'$ and $N'' = M''$.

Let ξ be a colouring of a cell D with a d-set M. For each $(u, v) \in M$, let

$$\begin{aligned} \lambda_\xi(u, v) &= 0, \quad \text{if} \quad \xi(u) = \xi(v), \\ &= 1, \quad \text{if} \quad \xi(u) \neq \xi(v). \\ \lambda_\xi(D) &= 0, \quad \text{if} \quad M = \varnothing, \\ &= \sum_{(u,v) \in M} \lambda_\xi(u, v), \quad \text{if} \quad M \neq \varnothing. \end{aligned}$$

TABLE I

$\lvert S(G)\rvert=1$	$\lvert S(G)\rvert=2$	$\lvert S(G)\rvert=3$	$\lvert S(G)\rvert=4$	$\lvert S(G)\rvert=5$	$\lvert S(G)\rvert=6$
	2 0 0	1 1 1	4 0 0	3 1 1	6 0 0
			2 2 0		4 2 0
					2 2 2

DEFINITION 3. A cell D is called even (odd) if for every realisable colouring the value $\lambda_\xi(D)$ is even (odd).

By definition only graphs of class 2 are even and odd cells simultaneously.

Now it is possible to formulate the first rule of construction of class 2 graphs.

RULE 1. A complete junction of two cells of different parities is a graph of class 2.

The following lemma was proved in [4, 5].

LEMMA 1. *Let G be a coloured graph and let p_i denote the number of semiedges of colour i $(i=1,2,3)$. Then*

$$p_1 \equiv p_2 \equiv p_3 \equiv |S(G)| \qquad (\text{mod } 2).$$

Proof. Let m_i be the number of edges of colour i $(i=1,2,3)$. Then $2m_i+p_i=|V(G)|$ so that $p_i \equiv |V(G)| \pmod 2$. Since $p_1+p_2+p_3=|S(G)|$, the result follows.

In Table I we list all triples satisfying the conditions $p_1 \equiv p_2 \equiv p_3 \equiv |S(G)| \pmod 2$, $p_1+p_2+p_3=|S(G)|$, and $p_1 \geqslant p_2 \geqslant p_3$ (for $|S(G)| \leqslant 6$).

By virtue of Table I, it is easy to prove the following:

LEMMA 2. *Let D be a cell such that* $\operatorname{rank}(D)=2$ *and* $|S(D)|=4$. *Then D is an even cell.*

LEMMA 3. *A junction of two cells of identical (resp. different) parities is an even (resp. odd) cell.*

The lemmas imply immediately a second rule of construction:

RULE 2. Let D be a junction of two cells D' and D'' of different parities. If $\operatorname{rank}(D)=2$ and $|S(D)|=4$ then $\operatorname{cl}(D)=2$.

A large set of even and odd cells may be obtained by using Lemma 3 and the following remarks:

I. Suppose that each pair of a d-set of a cell D consists of semiedges incident to the same vertex. It is evident that the parity of D is equal to the parity of rank(D). Both such cells, and cells described by Lemma 2, will be called trivial.

II. Let $(x_1, x_2, x_3, x_4, x_5, x_6)$ be a simple circuit, u_i be a semiedge incident to x_i $(i = 1, 2,..., 6)$. The cell which is generated by the circuit and the d-set $\{(u_1, u_4)(u_2, u_5)(u_3, u_6)\}$ will be denoted by C.

STATEMENT 1. *C is an even cell.*

Proof. Let ξ be a colouring of C, and let A_i denote the set of colours used for u_i and u_{i+3} $(i = 1, 2, 3)$. If $|A_i| = 1$ for some i then (see Lemma 2) $\lambda_\xi(C)$ is even. If two sets, for example A_1 and A_2, are the same, then (see Table I, column 6) $|A_3| = 1$ and consequently $\lambda_\xi(C)$ is even as well. So without loss of generality, we assume that $\xi(u_1) = 1$, $\xi(u_2) = 2$ and $A_2 = \{2, 3\}$. However, this yields a unique colouring of C and in it $\lambda_\xi(C)$ is even, which completes the proof.

III. The following operation may be applied only for cells D satisfying the conditions: rank$(D) = 3$, $|S(D)| = 6$. Note that the idea of the operation coincides with that of the Isaacs–Grinberg constructions.

Let $D_1, D_2,..., D_n$ be a sequence of cells such that $|S(D_i)| = 6$ $(i = 1,..., n)$, and let $\{u_{ij}, v_{ij}\}$ $(j = 1, 2, 3)$ be a d-set of D_i $(i = 1, 2,..., n)$. In the union $D_1 \cup D_2 \cup \cdots \cup D_n$ let us identify semiedges v_{ij} with $u_{i+1,j}$ $(j = 1, 2, 3;$ $i = 1,..., n-1)$. The resulting graph with the d-set $\{(u_{11}, v_{n3}), (u_{12}, v_{n2}), (u_{13}, v_{n1})\}$ will be called a hooking-cell of $D_1,..., D_n$.

STATEMENT 2. *The hooking-cell of an even number of odd cells is even.*

Proof. Let D be an odd cell such that $|S(D)| = 6$ and let $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ be a d-set of D. From Lemma 1 and the fact that D is odd, it follows that, for every colouring ξ of D, the matrix

$$\begin{vmatrix} \xi(u_1) & \xi(u_2) & \xi(u_3) \\ \xi(v_1) & \xi(v_2) & \xi(v_3) \end{vmatrix}$$

allows only four nonisomorphic realizations:

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}.$$

Note that the last two realizations are even permutations.

Now let D be a hooking-cell for $2k$ odd cells $D_1,..., D_{2k}$, let $\{(u_{ij}, v_{ij})\}$ $(i = 1, 2, 3)$ be a d-set of D_i $(i = 1,..., 2k)$ and let ξ be a colouring of D.

If any of the numbers 1, 2, 3 is used more than once in the first row of

TABLE II

$\xi(u_{11})$	$\xi(u_{12})$	$\xi(u_{13})$
$\xi(v_{11})$	$\xi(v_{12})$	$\xi(v_{13})$
$\xi(v_{21})$	$\xi(v_{22})$	$\xi(v_{23})$
...	...	...
$\xi(v_{n1})$	$\xi(v_{n2})$	$\xi(v_{n3})$

Table II, then the number is repeated twice in each odd row and therefore also in the last row of Table II. It follows that $\lambda_\xi(D)$ is even.

Suppose that in the first row no numbers are alike. Then any two rows of Table II represent an even permutation of the symbols 1, 2, 3. Hence the permutation

$$\begin{vmatrix} \xi(u_{11}) & \xi(u_{12}) & \xi(u_{13}) \\ \xi(v_{n1}) & \xi(v_{n2}) & \xi(v_{n3}) \end{vmatrix}$$

is realised in one of the following ways:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}.$$

In each case $\lambda_\xi(D) = 2$.

IV. Let G be a cubic graph, $x \in V(G)$, let x_1, x_2, x_3 be the vertices incident to x and let x_{i1} and x_{i2} be the vertices incident to x_1 and different from x_i $(i = 1, 2, 3)$. Let us remove the vertices x, x_1, x_2, x_3 from the graph G and consider the cell defined by the d-set $((x_{11}), (x_{12}))$, $((x_{21}), (x_{22}))$, $((x_{31}), (x_{32}))$. This cell will be denoted by $G[x]$.

STATEMENT 3. *If G is a regular graph of class 2 then $G[x]$ is an even cell for any x.*

Proof. If the statement is false and for some colouring ξ the value $\lambda = \lambda_\xi(G[x])$ is odd, then $\lambda = 3$, by Lemma 1. So $\xi(u_{12}) = \xi(u_{21}) = 2$, $\xi(u_{32}) = \xi(u_{23}) = 3$, $\xi(u_{11}) = \xi(u_{33}) = 1$. But in this case it is easy to extend ξ to a colouring of the whole graph, contradicting the fact that $\mathrm{cl}(G) = 2$.

It is easy to verify that in comparison with previously known methods the method of cells presents essentially more possibilities for the construction of class 2 graphs. However, we shall limit ourselves to showing only how the graphs of Petersen and Isaacs–Grinberg can be obtained.

For this purpose let us denote by Q the cell for which $V(G) = \{x_1, x_2, x_3, x\}$, $E(Q) = \{(x, x_1)(x, x_2)(x, x_3)\}$, $S(Q) = \{u_1, v_1, u_2, v_2, u_3, v_3\}$ and whose d-set is $\{(u_1, v_1)(u_2, v_2)(u_3, v_3)\}$, where u_i, v_i are semiedges incident to x_i $(i = 1, 2, 3)$.

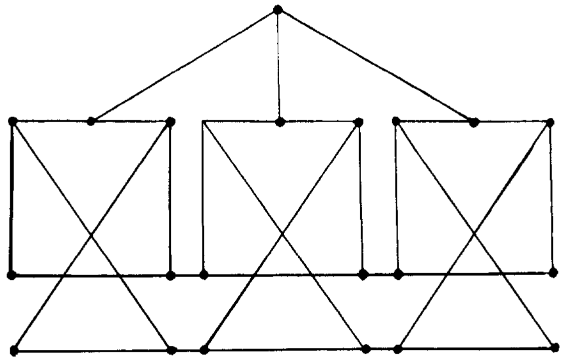

FIGURE 1

Now, the Petersen graph is the complete junction of C and Q; the graph I_n, for any $n \geqslant 1$, is the complete junction of Q and an even cell G_n which is the hooking-cell of $2n$ copies of Q.

4. Critical Graphs with an Even Number of Vertices

A connected graph G is called critical if it is of class 2 and if the removal of any edge of G lowers the chromatic index. In [14] Jakobsen had proved that there are no critical graphs of order 4, 6, 8 or 10. This fact let Jakobsen, and independently Beineke and Wilson [3], to formulate the following conjecture: every critical graph has an odd number of vertices. In [2] the conjecture was affirmed for graphs of order 14. Anderson and Fiorini (private communication) raised the bound to 16.

However, the conjecture is not true. Below, the sequence $\{F^n\}$ of irreducible critical graphs of even order will be constructed. F^1 is represented in Fig. 1. We now describe F^n for $n > 1$.

Let $C^1, C^2, ..., C^{(2n+1)}$ be $2n+1$ copies of C; $x_1^i, ..., x_6^i$ be a naturally ordered sequence of vertices of C^i; and u_k^i be a semiedge incident to x_k^i $(k = 1, ..., 6;\ i = 1, ..., 2n+1)$. Let us denote by K^{2n+1} an odd trivial cell with[1] $V(K^{2n+1}) = \{z_1, ..., z_{2n+1};\ y_1, ..., y_{2n+1}\}$, $E(K^{2n+1}) = \{(z_i, z_{i+1})(z_i, y_i)\}_{i=1}^{2n+1}$ and $S(K^{2n+1}) = \{v_i, w_i\}_{i=1}^{2n+1}$, where semiedges v_i and w_i are incident to the same vertex y_i $(i = 1, ..., 2n+1)$. In the union $C^1 \cup \cdots \cup C^{2n+1} \cup K^{2n+1}$ we shall identify semiedges u_6^i with u_1^{i+1}, u_3^{i+1} with u_4^{i+1} $(i = 1, ..., 2n)$ and then v_i with u_2^i, w_i with u_5^i $(i = 1, ..., 2n+1)$. The resulting graph is F^n.

STATEMENT 4. *The graph F^n is critical for any n.*

[1] z_{2n+2} is z_1.

Proof. It is clear that the conditions of Rule 2 are satisfied so that $\mathrm{cl}(F^n) = 2$. In order to prove that F^n is critical we must show that F^n_u is colourable for any $u \in E(F^n)$. This can be easily shown by considering the following simple facts:

(i) For any $i = 1,..., 2n$ there exists a colouring ξ of K^{2n+1} and a colouring η of $K^{2n+1}_{(z_i, z_{i+1})}$ for which

$$\begin{aligned} \xi(y_j, z_j) &= 1, && \text{if } j = i, \\ &= 2, && \text{if } j = i+1, \\ &= 3, && \text{if } j \neq i, i+1; \\ \eta(y_j, z_j) &= 3 && (j = 1,..., 2n+1). \end{aligned}$$

(ii) Let T be a graph obtained from C by adding a new vertex x_7; new edges (x_2, x_7), (x_5, x_7); and a semiedge (x_7). Then there exists a colouring ξ of T and for each $u \in E(T)$ a colouring η of T_u such that

$$\xi((x_1)) = \xi((x_4)) = \xi((x_7)) = 3, \qquad \xi((x_3)) = 1, \qquad \xi((x_6)) = 2;$$
$$\eta((x_1)) = \eta((x_4)) = \eta((x_6)) = 3, \qquad \eta((x_3)) = \eta((x_7)) = 1.$$

5. Description of Snarks with Cyclic Edge-Connectivity Equal to 4

A set of edges of a graph is called a cyclic edge-cut, or simply c-cut, if after its deletion the resulting graph contains at least two components having circuits.

A graph G having c-cuts is said to be cyclically s-edge-connected if each c-cut contains at least s edges. The minimal number of edges of a c-cut will be called the cyclic edge-connectivity and will be denoted by $z(G)$.

Let G and H be cubic graphs, let x and y be adjacent vertices of G, and let x_1, x_2 and y_1, y_2 be vertices adjacent to x and y, respectively. Also let $u = (p, q)$ and $v = (s, t)$ be edges of H. We remove the vertices x, y from H. In the union $G_{x,y} \cup H_{u,v}$ we now identify semiedges (x_1) with (p), (x_2) with (q), (y_1) with (s), and (y_2) with (t). The resulting graph will be denoted by $G \circ H$. The following result is shown in [1, 11].

THEOREM. *If* $\mathrm{cl}(G) = \mathrm{cl}(H) = 2$ *then* $\mathrm{cl}(G \circ H) = 2$ *also.*

Without going into details, we remark that the operation $\circ$ can easily be described in terms of the Hajós-union. For this purpose it is necessary to apply the Hajós-union to graphs G_y and H_v and to add two new edges to

TABLE III

e_1	e_2	e_3	e_4
1	1	1	1
1	1	2	2
1	2	1	2
1	2	2	1

achieve regularity. Thus, the theorem is a consequence of Jakobsen's results.

It is clear that $z(G \circ H) \leqslant 4$ always.

STATEMENT 5. *Let F be a snark,* $z(F) = 4$ *and after removing some c-cut of* 4*-edges all resulting components are of class* 1. *Then there exist cubic graphs G and H such that* $\mathrm{cl}(G) = \mathrm{cl}(H) = 2$ *and* $G \circ H = F$.

Proof. As $z(F) = 4$ we may suppose that a 4-edge c-cut $\{w_1, w_2, w_3, w_4\}$ exists such that if we remove its edges then exactly two components F' and F'' remain, each of which contains a circuit.

Let $w_i = (z_i', z_i'')$, where $z_i' \in F'$, $z_i'' \in F''$; denote by e_i the semiedge of F' incident to z_i' $(i = 1, 2, 3, 4)$.

There are four possibilities for colouring of semiedges e_1, e_2, e_3, e_4 (see Table III, column 4).

It is clear that the maximal $(1, 2)$-chain beginning with e_1 will end with another semiedge. Hence recolouring of this $(1, 2)$-chain substitutes the initial colouring of semiedges by another one. Thus at least two colourings of semiedges of F' are realisable by colourings of F'. The same is true for F''. But since $\mathrm{cl}(F) = 2$ the sets N' and N'' of realisable colourings of semiedges of F' and F'', respectively, are disjoint. Consequently, without loss of generality, we may assume that $N' = \{(1, 1, 1, 1)(1, 1, 2, 2)\}$, $N'' = \{(1, 2, 1, 2)(1, 2, 2, 1)\}$.

If we add to F' two new vertices x, y and edges (x, z_1'), (x, z_2'), (x, z_3'), (y, z_4'), (y, x) and then add to F'' two new edges (z_1'', z_2''), (z_3'', z_4'') we form two graphs H and G for which $G \circ H = F$ and $\mathrm{cl}(G) = \mathrm{cl}(H) = 2$. The statement is proved.

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REFERENCES

1. ADELSON-VELSKI AND A. TITOV, On 4-chromatic cubic graphs, *Vop. Kibern.* **1** (1974).
2. L. D. ANDERSEN, "Edge-Colourings of Simple and Non-simple Graphs," Aarhus University, 1975.
3. L. W. BEINEKE AND R. J. WILSON, On the edge-chromatic number of a graph, *Discrete Math.* **5** (1973), 15–20.
4. D. BLANUŠA, The problem of four colours, *Hrvat. Prirodosl. Drust. Glas. Mat. Fiz. Astron., Ser. II,* **1** (1946), 31–42.
5. B. DESCARTES, Network colourings, *Math. Gaz.* **32** (1948), 67–69.
6. S. FIORINI AND R. J. WILSON, Edge-colourings of graphs, *Res. Notes Math.* **16** (1977).
7. M. GARDNER, Mathematical games, *Sci. Amer.* **234**, No. 4 (1976), 126–130.
8. M. K. GOLDBERG, On graphs of degree 3 with chromatic index 4, *Bull. Acad. Sci. GSSR,* **93** (1979), 29–31.
9. M. K. GOLDBERG, Critical graphs with an even number of vertices, *Bull. Acad. Sci. GSSR,* **94** (1979), 25–27.
10. G. HAJÓS, Über eine Konstruction nicht n-färbarer Graphen, *Wiss. Z. Martin Luther Univ. Halle Wittenberg, Math. Naturwiss. Reihe* **10** (1961), 116, 117.
11. R. ISAACS, Infinite families of non-trivial trivalent graphs which are not Tait colorable, *Amer. Math. Monthly* **82** (1975), 221–239.
12. R. ISAACS, Loupekhine's snarks: A bifamily of non-Tait-colorable graphs, *J. Combin. Theory B,* in press.
13. I. T. JAKOBSEN, Some remarks on the chromatic index of a graph, *Arch. Math. (Basel)* **24** (1973), 440–448.
14. I. T. JAKOBSEN, On critical graphs with chromatic index 4, *Discrete Math.* **9** (1974), 265–276.
15. J. PETERSEN, Die Theorie regulären Graphen, *Acta Math.* **15** (1891), 193–220.
16. G. SZEKERES, Polyhedral decompositions of cubic graphs, *Bull. Aust. Math. Soc.* **8** (1973), 367–387.
17. V. G. VIZING, The chromatic class of a multigraph, *Kibernetika (Kiev)* **3** (1965), 29–39.